Algebra Preliminary Examination

June 2016

Answer all questions.

1. For any group G, define $G_1 = G$ and $G_{n+1} = (G, G_n)$. Call G *nilpotent* if $G_N = \{1\}$ for some N. Prove that any group of prime power order is nilpotent.

2. Let K be a field and G a finite group.

a) Prove that the number of one-dimensional K-representations of G is (up to equivalence) at most $[G : G']$.

b) Show with an example that the inequality can be strict.

3. Find all prime ideals of $\mathbf{F}_3[x, y]/(y^2 - x^3 + x)$ whose intersection with $\mathbf{F}_3[x]$ is equal to $(x^2 + x + 2)$.

4. a) Suppose R is a ring and $\phi : A \to B$ is a surjective homomorphism of left R-modules. Prove that for any right R-module M, $\phi \otimes \mathrm{id} : A \otimes_R M \to B \otimes_R M$ is also surjective.

b) Give an example to show that the above is false if both occurrences of "surjective" are replaced with "injective".

5. Suppose K/F is a normal algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

6. Find an explicit decomposition into direct product of matrix algebras over division rings of $M_2(\mathbf{F}_4) \otimes_{\mathbf{F}_2} M_2(\mathbf{F}_4)$.